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TRANSIENT AND TIME HARMONIC WAVES IN ELASTIC PLATES

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Abstract—This paper deals with transient wave propagation in elastic homogeneous and isotropic plates, in terms of displacement discontinuities of all order ≥ 1 at the wave front. The problem of steady state time harmonic waves is dealt with in terms of an asymptotic series expansion. The possible wave types along with the general transport-induction equations for each type are given, and the interrelationship between transient and time harmonic waves is discussed. Special constrained wave motions which allow uncoupling of the various possible wave types are defined. Several illustrative examples of the theory developed are given.

INTRODUCTION

In a recent paper Cohen[1] has treated the problem of wave propagation within the framework of the linear theory of homogeneous isotropic elastic plates. In that paper the problems of shock and acceleration waves were treated for arbitrarily shaped waves. Consideration was restricted to finding only the value of the disturbance at the wave front, or in other words to finding what is termed the geometric acoustics solution. In addition, the problem of steady state harmonic plane waves was also dealt with.

The aim of the present paper is three-fold. First, we wish to extend the geometric acoustics analysis so as to find values of the disturbance not only at the wave front but in a region behind it. This involves finding discontinuities of all order ≥ 1 at the wave front and representing the transient solution in terms of a series expansion behind the wave front. Such expansions are suggested in the monographs of Friedlander[2] and Achenbach[3]. Secondly, we desire to find steady state harmonic waves of a type more general than plane waves. Here, we employ the asymptotic series utilized by Karal and Keller[4], who dealt with such waves in three-dimensional linear elastic continua. Moreover, we wish to establish the one to one correspondence between transient and time harmonic waves in plates, a result analogous to that obtained by Kline and Kay[5] for the electromagnetic field equations. Finally, we desire to illustrate the results of these analyses, by applying them to problems of a specific nature.

1. THE PLATE EQUATIONS

We consider the propagation of waves in linear, isotropic and homogeneous elastic plates. The plate equations that we utilize are those of linearised Cosserat plate theory as developed by Naghdi[6]. These equations developed from a direct two-dimensional approach are based on a director model and are equivalent to those developed from three-dimensional considerations, and include the effects of transverse shear deformation, transverse normal stress and strain and rotatory inertia. The displacement equations of motion separate into two sets governing the extensional and bending motions, respectively[1]. These are

$$\mu\nabla^2\mathbf{u} + (\lambda+\mu)\nabla(\nabla\cdot\mathbf{u}) + \lambda\nabla\delta^3 + \rho h^{-1}\mathbf{F} = \rho h^{-1}\ddot{\mathbf{u}}, \tag{1.1}$$

$$\alpha_8\nabla^2\delta^3 - (\lambda+2\mu)h\delta^3 - \lambda(\nabla\cdot\mathbf{u})h + \rho L^3 = \rho\alpha\ddot{\delta}^3, \tag{1.2}$$

for the extensional theory, and

$$\mu\nabla^2\boldsymbol{\delta} + \mu(3\lambda+2\mu)(\lambda+2\mu)^{-1}\nabla(\nabla\cdot\boldsymbol{\delta}) - \{\alpha_3(\boldsymbol{\delta}+\nabla u^3) + \rho\mathbf{L}\}/h\alpha = \rho h^{-1}\ddot{\boldsymbol{\delta}}, \tag{1.3}$$

$$\alpha_3(\nabla\cdot\boldsymbol{\delta} + \nabla^2 u^3) + \rho F^3 = \rho\ddot{u}^3, \tag{1.4}$$

for the bending theory.

In the above equations $\mathbf{u}$, u^3 and $\boldsymbol{\delta}$, δ^3 denote the displacements of the Cosserat plane and of the director, respectively. The vectors $\mathbf{u}$, $\boldsymbol{\delta}$ represent the displacements parallel to the plane of the plate, while u^3, δ^3 represent displacements normal to the plate. For convenience we define $\mathbf{w}_1 = (\mathbf{u}, \delta^3)$, $\mathbf{w}_2 = (\boldsymbol{\delta}, u_3)$, so that $\mathbf{w}_1$ and $\mathbf{w}_2$ are displacement vectors associated with extensional and bending motions, respectively. ∇ is the two-dimensional gradient operator in the plane of the plate. Also λ, μ are Lame's constants, $\mathbf{F}$, F^3 are body forces, $\mathbf{L}$, L^3 are body couples, $\alpha = h^2/12$ and α_3, α_8 are constant constitutive coefficients. The mass per unit area is ρ while h is the plate thickness.

2. TRANSIENT WAVES AND DISCONTINUITIES

Consider a source of disturbance acting over some curve in a homogeneous isotropic elastic plate.† If the source begins to act at time $t = 0$, then for $t > 0$ this disturbance will spread into the plate with a constant wave front velocity G. The wave front will constitute a family of parallel curves $\psi(x, y) = Gt$ in the x, y plane of the plate while sweeping out a hypercone $\phi(x, y, t) = 0$ in space-time. The value of the field at a point $P_0(x_0, y_0, t_0)$ on the wave front is called the geometrical acoustic field by analogy to the geometrical optics situation arising in [5]. The value of this field at any point $P(x_0, y_0, t)$, $t > t_0$, behind the wave front will constitute the so-called transient solution to the disturbance problem.

We assume a transient solution to eqns (1.1)–(1.4) in the form of a Taylor's series expansion[2, 3] at the wavefront into the region behind it. Thus we write

$$\mathbf{w}_\alpha = \sum_{n=1}^{\infty} [\partial^n \mathbf{w}_\alpha/\partial t^n]_{t=t_0} \frac{\langle t - t_0 \rangle^n}{n!}, \tag{2.1}$$

where $\langle , \rangle = 0$ if the argument is negative while [ ] indicates the discontinuity or jump of the argument across the wave front. These discontinuities occur at the wave front since the region ahead of the wave is undisturbed. The wave is thus naturally a carrier of discontinuities. The lowest order derivative of $\mathbf{w}_\alpha$ having a discontinuity defines the order of the wave. A first order wave is called a shock or strain wave and waves of this type will constitute the subject matter dealt with herein. Higher order waves yield results which are completely analogous to those for first order waves. For first order waves, a knowledge of the first order discontinuities on the wave front will constitute the geometric acoustics solution, while a knowledge of the higher order discontinuities will allow calculation of the transient solution from eqn (2.1).

Associated with the geometry of the wave front at any point are its unit tangent $\boldsymbol{\lambda}$ and unit normal $\boldsymbol{\nu}$. We shall employ the notation $w_\alpha{}^\lambda = \mathbf{w}_\alpha \cdot \boldsymbol{\lambda}$ and $w_\alpha{}^\nu = \mathbf{w}_\alpha \cdot \boldsymbol{\nu}$. Moreover, we use l and s to denote arc lengths along the wave curves and rays, respectively, with the corresponding directional derivatives defined by $\mathrm{d}/\mathrm{d}l = \boldsymbol{\lambda} \cdot \nabla$ and $\mathrm{d}/\mathrm{d}s = \boldsymbol{\nu} \cdot \nabla$. In order to deal with discontinuities we shall utilize the compatibility equations

$$[\mathbf{w}_{\alpha,n+1}] = -G[\mathrm{d}\mathbf{w}_{\alpha,n}/\mathrm{d}s] + \frac{D}{Dt}[\mathbf{w}_{\alpha,n}], \qquad n \geq 0, \tag{2.2}$$

where $\mathbf{w}_{\alpha,n} = \partial^n \mathbf{w}_\alpha/\partial t^n$ and D/Dt denotes time differentiation as seen by an observer moving with the wave.‡

We calculate the $(n-1)$st order derivative, $n \geq 1$, of the equations of motion (1.1)–(1.4) and take the jump of the resulting equations. On eliminating explicit dependence on t through the equation $t = \psi/G$ and assuming C^∞ body forces and couples, we find after considerable manipulation that

$$(\mu - G^2\rho h^{-1})\mathbf{a}_{n+1} + (\lambda + \mu)\{\boldsymbol{\nu}(\boldsymbol{\nu} \cdot \mathbf{a}_{n+1} - G\nabla \cdot \mathbf{a}_n) - G\nabla(\boldsymbol{\nu} \cdot \mathbf{a}_n - G\nabla \cdot \mathbf{a}_{n-1})\}$$
$$+ \mu G\{\kappa \mathbf{a}_n - 2\boldsymbol{\nu} \cdot \nabla \mathbf{a}_n + G\nabla^2 \mathbf{a}_{n-1}\} - \lambda G\{\boldsymbol{\nu} a_n{}^3 - G\nabla a^3_{n-1}\} = \mathbf{0}, \tag{2.3}$$

†The results to follow are readily generalized to nonhomogeneous plates. The general features of the analysis are analogous to those presented here. The complication appears as an algebraic one, due to the fact that the speed of propagation is no longer constant.

‡We refer the reader to Thomas[7] for a general discussion of compatibility relations and waves.

$$(\alpha_8 - G^2\rho\alpha)a^3_{n+1} + \alpha_8 G\{\kappa a_n^3 - 2\boldsymbol{\nu}\cdot\nabla a_n^3 + G\nabla^2 a^3_{n-1}\} + \lambda h G\{\boldsymbol{\nu}\cdot\mathbf{a}_n - G\nabla\cdot\mathbf{a}_{n-1}\} - (\lambda+2\mu)G^2 a^3_{n-1} = 0, \quad (2.4)$$

$$(\mu - G^2\rho h^{-1})\mathbf{b}_{n+1} + \frac{(3\lambda+2\mu)\mu}{(\lambda+2\mu)}\{\boldsymbol{\nu}(\boldsymbol{\nu}\cdot\mathbf{b}_{n+1} - G\nabla\cdot\mathbf{b}_n) - G\nabla(\boldsymbol{\nu}\cdot\mathbf{b}_n - G\nabla\cdot\mathbf{b}_{n-1})\}$$
$$+\mu G\{\kappa\mathbf{b}_n - 2\boldsymbol{\nu}\ \ \nabla\mathbf{b}_n + G\nabla^2\mathbf{b}_{n-1}\} + \alpha_3 G(h\alpha)^{-1}\{\boldsymbol{\nu}b_n^3 - G(\mathbf{b}_{n-1} + \nabla b^3_{n-1})\} = 0, \quad (2.5)$$

$$(\alpha_3 - G^2\rho)b^3_{n+1} + \alpha_3 G\{\kappa b_n^3 - \boldsymbol{\nu}\cdot(2\nabla b_n^3 + \mathbf{b}_n) + G(\nabla\cdot\mathbf{b}_{n-1} + \nabla^2 b^3_{n-1})\} = 0, \quad (2.6)$$

where $\kappa = -\nabla^2\psi$ is the curvature of the wave front and we have set

$$\mathbf{a}_n = [\mathbf{u}_{,n}], \qquad a_n^3 = [\delta^3_{,n}], \qquad \mathbf{b}_n = [\boldsymbol{\delta}_{,n}], \qquad b_n^3 = [u^3_{,n}]. \quad (2.7)$$

Equations (2.3)–(2.6) hold for $n \geq 1$ as a consequence of eqns (1.1)–(1.4) and also for the case $n = 0$ as a consequence of the form of eqns (1.1)–(1.4) at the curve of discontinuity. This latter case is fully treated in [1].

If we take the scalar product of the vector equations (2.3) and (2.5) with $\boldsymbol{\lambda}$ and $\boldsymbol{\nu}$, then the set of equations (2.3)–(2.6) yield the following classification of shock waves, along with their transport-induction equations governing jumps of all order, $n \geq 1$.

(i) *Longitudinal wave*

$$a_1^\nu \neq 0, \qquad a_1^\lambda = a_1^3 = 0, \qquad G_L^2 = (\lambda+2\mu)h/\rho.$$

$$\frac{\delta a_n^\nu}{\delta s} = -\frac{1}{2(1-\nu)}\left\{\frac{\mathrm{d}a_n^\lambda}{\mathrm{d}l} - G_L\frac{\mathrm{d}}{\mathrm{d}s}\nabla\cdot\mathbf{a}_{n-1} + 2\nu\left(a_n^3 - G_L\frac{\mathrm{d}a^3_{n-1}}{\mathrm{d}s}\right)\right\} + G_T^2 G_L^{-1}\boldsymbol{\nu}\cdot\nabla^2\mathbf{a}_{n-1}, \quad (2.8)$$

$$a^\lambda_{n+1} = -(1-2\nu)G_L\left\{\frac{\delta a_n^\lambda}{\delta s} - G_L\boldsymbol{\lambda}\cdot\nabla^2\mathbf{a}_{n-1}\right\} + G_L\frac{\mathrm{d}}{\mathrm{d}l}\{G_L(\nabla\cdot\mathbf{a}_{n-1} + 2\nu a^3_{n-1}) - a_n^\nu\}, \quad (2.9)$$

$$a^3_{n+1} = (G_L^2 - G_S^2)^{-1}\left\{\frac{2\nu\alpha^{-1}}{(1-2\nu)}G_T^2 G_L(a_n^\nu - G_L\nabla\cdot\mathbf{a}_{n-1}) - G_L G_S^2\left(\frac{\delta a_n^3}{\delta s} - G_L\nabla^2 a_n^3\right) - \alpha^{-1}G_L^4 a^3_{n-1}\right\}. \quad (2.10)$$

(ii) *Shear wave*

$$a_1^\lambda \neq 0, \qquad a_1^\nu = a_1^3 = 0, \qquad G_T^2 = \mu h/\rho.$$

$$\frac{\delta a_n^\lambda}{\delta s} = -\frac{1}{(1-2\nu)}\frac{\mathrm{d}}{\mathrm{d}l}\{a_n^\nu - G_T\nabla\cdot\mathbf{a}_{n-1}\} - G_T\{2\nu a^3_{n-1} + \boldsymbol{\lambda}\cdot\nabla^2\mathbf{a}_{n-1}\}, \quad (2.11)$$

$$a^\nu_{n+1} = G_T\left\{\frac{\mathrm{d}a_n^\lambda}{\mathrm{d}l} + 2(1-\nu)\frac{\delta a_n^\nu}{\delta s} - (1-2\nu)G_T\boldsymbol{\nu}\cdot\nabla^2\mathbf{a}_{n-1} + 2\nu\left(a_n^3 - G_T\frac{\mathrm{d}a^3_{n-1}}{\mathrm{d}s}\right) - G_T\frac{\mathrm{d}}{\mathrm{d}s}\nabla\cdot\mathbf{a}_{n-1}\right\}, \quad (2.12)$$

$$a^3_{n+1} = (1 - G_S^2/G_T^2)^{-1}\left\{\frac{2\nu\alpha^{-1}}{(1-2\nu)}G_T(a_n^\nu - G_T\nabla\cdot\mathbf{a}_{n-1}) - G_S^2 G_T^{-1}\frac{\delta a_n^3}{\delta s} + G_S^2\nabla^2 a^3_{n-1} - \alpha^{-1}G_L^2 a^3_{n-1}\right\}. \quad (2.13)$$

(iii) *Squeeze-gradient wave*

$$a_1^3 \neq 0, \qquad \mathbf{a}_1 = \mathbf{0}, \qquad G_S^2 = \alpha_8/\rho\alpha.$$

$$\frac{\delta a_n^3}{\delta s} = \frac{2\nu\alpha^{-1}}{(1-2\nu)}G_T^2 G_S^{-2}\{a_n^\nu - G_S\nabla\cdot\mathbf{a}_{n-1}\} + G_S\nabla^2 a^3_{n-1} - \alpha^{-1}G_L^2 G_S^{-2}a^3_{n-1}, \quad (2.14)$$

$$a^\nu_{n+1} = (G_L^2 - G_S^2)^{-1}G_S\left\{G_L^2\frac{\delta a_n^\nu}{\delta s} - G_T^2 G_S\boldsymbol{\nu}\cdot\nabla^2\mathbf{a}_{n-1} + \frac{1}{2(1-\nu)}G_L^2\left(\frac{\mathrm{d}a_n^\lambda}{\mathrm{d}l} - G_S\frac{\mathrm{d}}{\mathrm{d}s}\nabla\cdot\mathbf{a}_{n-1}\right)\right.$$
$$\left. + \frac{2\nu}{(1-2\nu)}G_T^2\left(a_n^3 - G_S\frac{\mathrm{d}a^3_{n-1}}{\mathrm{d}s}\right)\right\}, \quad (2.15)$$

$$a_{n+1}^{\lambda}=(G_T^2-G_S^2)^{-1}G_S\left\{G_T^2\left(\frac{\delta a_n^{\lambda}}{\delta s}-G_S\boldsymbol{\lambda}\cdot\nabla^2\mathbf{a}_{n-1}\right)+\frac{1}{2(1-\nu)}\frac{\mathrm{d}}{\mathrm{d}l}\left[G_L^2(a_n^{\nu}-G_S\nabla\cdot\mathbf{a}_{n-1})\right.\right.$$

$$\left.\left.-\frac{2\nu}{(1-2\nu)}G_T^2G_Sa_{n-1}^3\right]\right\}. \tag{2.16}$$

(iv) *Bending wave*

$$b_1^{\nu}\neq 0,\qquad b_1^{\lambda}=b_1^3=0,\qquad G_B^2=4\mu(\lambda+\mu)h/(\lambda+2\mu)\rho.$$

$$\frac{\delta b_n^{\nu}}{\delta s}=-\frac{(1+\nu)}{2}\left\{\frac{\mathrm{d}b_n^{\lambda}}{\mathrm{d}l}-G_B\frac{\mathrm{d}}{\mathrm{d}s}\nabla\cdot\mathbf{b}_{n-1}\right\}+G_B^{-1}\left\{G_T^2\boldsymbol{\nu}\cdot\nabla^2\mathbf{b}_{n-1}-\alpha^{-1}G_K^2\right.$$

$$\left.\times\left(b_{n-1}^{\nu}+G_B^{-1}b_n^3+\frac{\mathrm{d}b_{n-1}^3}{\mathrm{d}s}\right)\right\}, \tag{2.17}$$

$$b_{n+1}^{\lambda}=-(1-G_T^2/G_B^2)^{-1}\left\{G_T^2\left[G_B^{-1}\frac{\delta b_n^{\lambda}}{\delta s}-\boldsymbol{\lambda}\cdot\nabla^2\mathbf{b}_{n-1}+\frac{(1+\nu)}{(1-\nu)}\frac{\mathrm{d}}{\mathrm{d}l}\left(G_B^{-1}b_n^{\nu}-\nabla\cdot\mathbf{b}_{n-1}\right)\right]\right.$$

$$\left.+\alpha^{-1}G_K^2\left(b_{n-1}^{\lambda}+\frac{\mathrm{d}b_{n-1}^3}{\mathrm{d}l}\right)\right\}, \tag{2.18}$$

$$b_{n+1}^3=(1-G_B^2/G_K^2)^{-1}G_B\left\{\frac{\delta b_n^3}{\delta s}+b_n^{\nu}-G_B(\nabla\cdot\mathbf{b}_{n-1}+\nabla^2b_{n-1}^3)\right\}. \tag{2.19}$$

(v) *Twisting wave*

$$b_1^{\lambda}\neq 0,\qquad b_1^{\nu}=b_1^3=0,\qquad G_T^2=\mu h/\rho.$$

$$\frac{\delta b_n^{\lambda}}{\delta s}=-\frac{(1+\nu)}{(1-\nu)}\frac{\mathrm{d}}{\mathrm{d}l}\{b_n^{\nu}-G_B\nabla\cdot\mathbf{b}_{n-1}\}+G_T\boldsymbol{\lambda}\cdot\nabla^2\mathbf{b}_{n-1}-\alpha^{-1}G_K^2G_T^{-1}\left\{b_{n-1}^{\lambda}+\frac{\mathrm{d}b_{n-1}^3}{\mathrm{d}l}\right\}, \tag{2.20}$$

$$b_{n+1}^{\nu}=(G_B^2-G_T^2)^{-1}G_T\left\{G_B^2\frac{\delta b_n^{\nu}}{\delta s}+\alpha^{-1}G_K^2\left[b_n^3+G_T\left(b_{n-1}^{\nu}+\frac{\mathrm{d}b_{n-1}^3}{\mathrm{d}s}\right)\right]\right.$$

$$\left.+\frac{(1+\nu)}{(1-\nu)}G_T^2\left(\frac{\mathrm{d}b_n^{\lambda}}{\mathrm{d}l}-G_T\frac{\mathrm{d}}{\mathrm{d}s}\nabla\cdot\mathbf{b}_{n-1}\right)-G_T^3\boldsymbol{\nu}\cdot\nabla^2\mathbf{b}_{n-1}\right\}, \tag{2.21}$$

$$b_{n+1}^3=(1-G_T^2/G_K^2)^{-1}G_T\left\{\frac{\delta b_n^3}{\delta s}+b_n^{\nu}-G_T(\nabla\cdot\mathbf{b}_{n-1}+\nabla^2b_{n-1}^3)\right\}. \tag{2.22}$$

(vi) *Kink wave*

$$b_1^3\neq 0,\qquad \mathbf{b}_1=\mathbf{0},\qquad G_K^2=\alpha_3/\rho.$$

$$\frac{\delta b_n^3}{\delta s}=G_K\{\nabla\cdot\mathbf{b}_{n-1}+\nabla^2b_{n-1}^3\}-b_n^{\nu}, \tag{2.23}$$

$$b_{n+1}^{\nu}=(G_B^2-G_K^2)^{-1}G_K\left\{G_B^2\frac{\delta b_n^{\nu}}{\delta s}+\frac{(1+\nu)}{(1-\nu)}G_T^2\left(\frac{\mathrm{d}b_n^{\lambda}}{\mathrm{d}l}-G_K\frac{\mathrm{d}}{\mathrm{d}s}\nabla\cdot\mathbf{b}_{n-1}\right)\right.$$

$$\left.-G_KG_T^2\boldsymbol{\nu}\cdot\nabla^2\mathbf{b}_{n-1}+\alpha^{-1}G_K^3\left(b_{n-1}^{\nu}+\frac{\mathrm{d}b_{n-1}^3}{\mathrm{d}s}-G_K^{-1}b_n^3\right)\right\}, \tag{2.24}$$

$$b_{n+1}^{\lambda}=(1-G_K^2/G_T^2)^{-1}G_K\left\{\frac{\delta b_n^{\lambda}}{\delta s}+G_K\left[\boldsymbol{\lambda}\cdot\nabla^2\mathbf{b}_{n-1}+\frac{(1+\nu)}{(1-\nu)}\frac{\mathrm{d}}{\mathrm{d}l}(G_K^{-1}b_n^{\nu}-\nabla\cdot\mathbf{b}_{n-1}\right]\right.$$

$$\left.-\alpha^{-1}G_K^3G_T^{-2}\left(b_{n-1}^{\lambda}+\frac{\mathrm{d}b_{n-1}^3}{\mathrm{d}l}\right)\right\}. \tag{2.25}$$

In the above equations ν is Poisson's ratio and we have introduced the operator $\delta/\delta s = 2(\mathrm{d}/\mathrm{d}s)-\kappa$. The first equation in each set is the transport or growth-decay equation and it provides a differential equation for the variation of the jumps along the rays. These equations are all of the same form and their solution may be found in [3] or [4]. The pairs of algebraic equations following the transport equation determine the induced higher order discontinuities and bring into play the coupling of wave types. The results obtained are a generalization of those presented in [1].

3. STEADY STATE TIME HARMONIC WAVES

In [1] the question of steady state time harmonic plane wave solutions of the plate equations (1.1)–(1.4) in the absence of body forces and couples was examined. For waves of this type the possible phase velocities $V = \omega k^{-1}$, where ω is the frequency and k the wave number, were found as a function of wave number. It was found that with the exception of one mode of propagation in the case of extensional waves, that all other wave types were dispersive. In the limiting case of infinite frequency or wave number, all phase velocities reduced to the corresponding speeds of propagation of discontinuities given in Section 2 of this paper. Here we seek harmonic wave solutions of a more general nature than those in [1], corresponding to waves which are generally curved and which arise as a consequence of a time harmonic disturbance applied to an arbitrary curve in the plane of the plate.

Thus we begin by assuming an asymptotic series for the displacements in the form

$$\mathbf{w}_\alpha = e^{i\omega(S-t)} \sum_{n=1}^{\infty} \frac{\mathbf{A}_{\alpha n}}{(i\omega)^n}, \tag{3.1}$$

which is to represent the steady state behaviour of the plate for large frequencies. Series of this type were introduced in [4], where S is called the phase function, to investigate steady state time harmonic behaviour in an unbounded three-dimensional elastic non-dispersive medium. For high frequencies the first term in the series predominates and we may regard this as an approximation to the solution. For other frequencies the higher order terms in the series may be viewed as corrections to the disturbance arising due to (a) the dispersive nature of the governing equations, (b) the geometry of the wave being non-planar, and (c) the variation of amplitude over the wave.

On substituting eqn (3.1) into eqns (1.1)–(1.4) in the absence of body forces, we arrive at a set of differential recurrence relations which are precisely those given by eqns (2.3)–(2.6) provided we make the identifications

$$\mathbf{A}_{\alpha n} = (-1)^n[\mathbf{w}_{\alpha,n}], \qquad S = G^{-1}\psi. \tag{3.2}$$

Thus the wave classification and transport-induction equations associated with steady state time harmonic waves as given by eqn (3.1) are in one to one correspondence with those pertaining to the propagation of transients as given by eqn (2.1). We note that the curves of constant phase correspond to the wave fronts in the transient problem. Moreover, the leading term in eqn (3.1) decays as the geometrical acoustic solution and may be regarded as representing a decaying harmonic wave whose phase velocity equals the wave front velocity.

Thus, as discussed in [5], we see from the aforementioned correspondence that the solution $\mathbf{w}_\alpha^{\omega}$ due to a time harmonic boundary condition $e^{-i\omega t}$ corresponds to a unit pulse solution $\mathbf{w}_\alpha^{H}$ due to a boundary condition involving $H(t)$.† The solution due to an arbitrary time dependent boundary condition $f(t)$, can be obtained through the Duhamel integral [8] as

$$\mathbf{w}_\alpha = \frac{\partial}{\partial t}\int_{[0,t]} \mathbf{w}_\alpha^{H}(t-\tau)f(\tau). \tag{3.3}$$

4. UNCOUPLED WAVE MOTIONS

In general, for each of the classes of extensional and bending waves, coupling will occur between the various wave types. As seen in [1] this is true even in the case of plane waves, where coupling occurs between the longitudinal and squeeze gradient waves within the framework of the extensional theory and between the bending and kink waves within the

†$H(t)$ denotes the Heaviside function and is defined by $H(t) = 0$, $t < 0$, and $H(t) = 1$, $t > 0$. It is related to the Dirac delta function by $\Delta(t) = \dot{H}(t)$, where the differentiation is in the generalised sense [2].

framework of the bending theory. The shear and twisting waves were uncoupled and might be referred to as pure waves. Our objective here is to see if it is possible to expand the category of pure plane waves. This in fact can be done by introducing suitable constrained motions, for which the constraints are produced by application of appropriate body forces and couples.

(i) *Pure tilting and twisting waves*

In eqns (1.3) and (1.4) we assume $\mathbf{w}_2 = \mathbf{w}_2(x, t)$, $\boldsymbol{\delta} = \delta^1\mathbf{i} + \delta^2\mathbf{j}$, $u^3 = 0$, $\mathbf{L} = 0$, where $\mathbf{i}$, $\mathbf{j}$ are unit vectors along the rectangular cartesian coordinate axes x, y. We obtain,

$$\frac{\partial^2\delta^1}{\partial t^2} - G_B{}^2\frac{\partial^2\delta^1}{\partial x^2} + \frac{G_K{}^2}{\alpha}\delta^1 = 0, \qquad F^3 = -\frac{\alpha_3}{\rho}\frac{\partial\delta^1}{\partial x}, \tag{4.1}$$

$$\frac{\partial^2\delta^2}{\partial t^2} - G_T{}^2\frac{\partial^2\delta^2}{\partial x^2} + \frac{G_K{}^2}{\alpha}\delta^2 = 0. \tag{4.2}$$

Equation $(4.1)_1$ defining a pure tilting† wave corresponds to a tilting of the plate cross section and requires a constraining body force given by eqn $(4.1)_2$. This constraint can be maintained by sandwiching the plate between two rigid layers. Equation (4.2) defines a pure twisting wave and its existence requires no constraint.

(ii) *Pure shear and squeeze gradient waves*

In equations (1.1) and (1.2) we assume $\mathbf{w}_1 = \mathbf{w}_1(x, t)$, $\mathbf{u} = \mathbf{u}^1 i + \mathbf{u}^2 j$, $u^1 = 0$, $L^3 = 0$, $\mathbf{F} = F^l\mathbf{i}$ and find that these conditions are satisfied provided

$$\frac{\partial^2\delta^3}{\partial t^2} - G_S{}^2\frac{\partial^2\delta^3}{\partial x^2} + \frac{G_L{}^2}{\alpha}\delta^3 = 0, \qquad F^1 = -\frac{\lambda h}{\rho}\frac{\partial\delta^3}{\partial x}, \tag{4.3}$$

$$\frac{\partial^2 u^2}{\partial t^2} - G_T{}^2\frac{\partial^2 u^2}{\partial x^2} = 0. \tag{4.4}$$

Equation (4.3) defines a pure squeeze gradient wave and requires that the plate midsurface be made inextensible. Equation (4.4) governs the propagation of pure shear waves and their existence requires no constraint.

(iii) *Pure kink waves*

In eqns (1.3) and (1.4) we now assume $\mathbf{w}_2 = \mathbf{w}_2(x, t)$, $F^3 = -(K/\rho)u^3$, $K \geq 0$, $\boldsymbol{\delta} = 0$, $L = L^1\mathbf{i}$ and we obtain the governing equation of a plane kink wave as

$$\frac{\partial^2 u^3}{\partial t^2} - G_K{}^2\frac{\partial^2 u^3}{\partial x^2} + KG_K{}^2u^3 = 0, \qquad L^1 = G_K{}^2\frac{\partial u^3}{\partial x}. \tag{4.5}$$

In order that this wave propagate, a constraining couple L^1 must be applied in order to assure that normals to the plane of the plate are constrained to remain normal. In the case $K \neq 0$, the problem corresponds to a plate on an elastic foundation with modulus K.

We note that each of the waves defined by eqns (4.1)–(4.3) and (4.5) satisfy the same differential equation and some form of constraining equation and hence it is only necessary to deal with one of these in order to solve them all. The governing transport equation‡ may be obtained by making the appropriate substitutions in the appropriate general forms of these in Section 2, or by directly seeking a solution to eqn $(4.1)_1$ in the form (3.1). An example will be considered in the next section.

The pure shear and twisting plane waves defined by eqns (4.4) and (4.2) may be generalised. By examining the induction equations (2.12), (2.13) and (2.21), (2.22), which correspond to these

†Logically we should call this wave a pure bending wave, but since this terminology usually has another meaning and since the terminology tilting is descriptive, we introduce it here.

‡Since there is no coupling, the induction equations make no contribution to the analysis. They are replaced by the appropriate constraint equations. For the case $K \neq 0$, the transport equation (2.23) for the kink wave does not apply, as it was derived on the basis of C^∞ body forces.

two types of waves, we see that no coupling will exist provided the wave discontinuities are constant along the wave fronts. From the form of the transport equations we see that this will be true only if the wave curves are of constant curvature, i.e. circular and if the discontinuities are constant on the initial wave curve. Hence we can have pure shear and twisting waves with circular wave fronts.

5. EXAMPLES

(i) *Torsional shear waves*

We consider the problem of an unbounded plate having a circular cavity of radius a and subjected to a uniform shear stress which is suddenly applied and maintained. Since the wavefronts are circular we employ polar coordinates (r, θ) and in eqns (1.1), (1.2) we assume

$$\mathbf{u} = u(r)\boldsymbol{\lambda}, \qquad \delta^3 = 0, \qquad \mathbf{F} = 0, \qquad L^3 = 0. \tag{5.1}$$

The appropriate boundary condition for this axi-symmetric problem is

$$\tau_{r\theta} = \mu\left(\frac{\partial u}{\partial r} - \frac{u}{r}\right) = \tau_0 H(t), \qquad \text{at } r = a, \tag{5.2}$$

where τ_0 is a constant. Using eqns (2.1), (2.11) and (5.2) we obtain

$$\begin{aligned} a_1^{\lambda} &= -\frac{\tau_0 G_T}{\mu}\left(\frac{a}{r}\right)^{1/2}, \\ a_2^{\lambda} &= \frac{3\tau_0 G_T^2}{8\mu a}\left\{5\left(\frac{a}{r}\right)^{1/2} - \left(\frac{a}{r}\right)^{3/2}\right\}, \\ a_3^{\lambda} &= -\frac{15\tau_0 G_T^3}{128\mu a^2}\left\{23\left(\frac{a}{r}\right)^{1/2} - 6\left(\frac{a}{r}\right)^{3/2} - \left(\frac{a}{r}\right)^{5/2}\right\}, \qquad \text{etc.} \end{aligned} \tag{5.3}$$

The solution to the transient problem is given by eqns (2.1) and (5.3) with $t_0 = (r-a)/G_T$ and then the shear stress is determined from eqn $(5.2)_1$. Goodier and Jahsman[9] used Laplace transforms to solve the above problem while Achenbach[3] treated the same problem via discontinuity analysis but proceeded along somewhat different lines than that employed here.

Experience in the numerical evaluation of transient solutions shows that these series solutions (2.1) converge slowly in certain cases. Turchetti and Mainardi[10] introduced Padé† approximants to accelerate the convergence of such series solutions. We have devised a simple numerical superposition technique as an alternate means of overcoming the same difficulty and have verified that the numerical results agree with those of [10]. This technique is based on the fact that if any physical system is subjected to a step boundary pulse of small duration t_0^*, the response must decay to zero after a definite time. By choosing a suitable value of t_0^*, it is possible in most cases to obtain a solution that approximately decays to zero by using a few terms in the series obtained from eqn (3.3). The boundary data is then subdivided into equivalent step pulses of duration t_0^* and the response due to each of these pulses, acting at the appropriate time, are superposed to obtain the required transient solution. For further details on this technique and its applications we refer the reader to [11].

In order to determine the shear stress by the superposition technique we use eqns (2.1), (3.3), (5.2) and (5.3). The results obtained without and with the incorporation of the superposition technique are called "series" and "modified" solutions respectively. In Fig. 1 we show our results obtained with $t_0^* = 0.1G_Tt/a$. The figure shows the excellent agreement between our modified solution and the closed form solution of [9].

(ii) *Pure tilting waves*

In this example we wish to illustrate the class of constrained waves discussed in Section 4 and the relationship between the unit pulse and the corresponding time harmonic problems. We

†We wish to thank Prof. J. D. Achenbach for calling to our attention the work on Padé approximants.

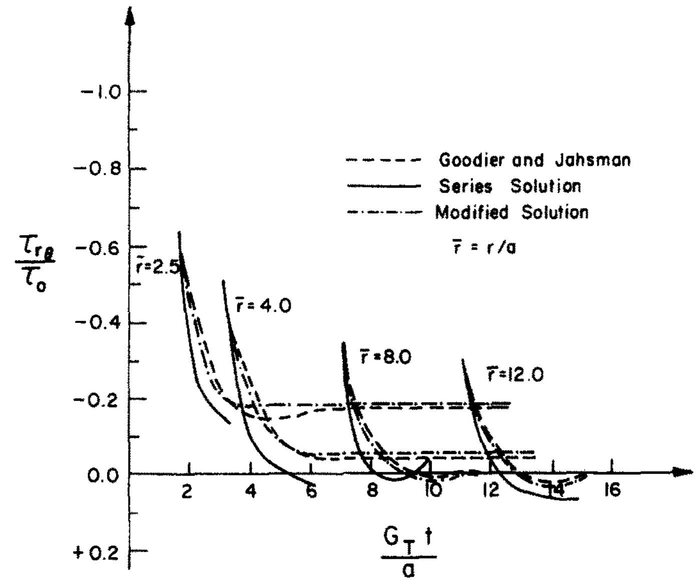

Fig. 1. Distribution of shear stress in the plate.

consider eqn $(4.1)_1$, subjected to either one of the following boundary conditions at $x = 0$,

$$\frac{\partial \delta^1}{\partial x} = e^{-i\omega t}, \qquad \frac{\partial \delta^1}{\partial x} = H(t). \tag{5.4}$$

Setting $w_2 = \delta\nu$, $A_{2n} = (-1)^n b_n{}^\nu \nu$ and using eqns (2.17), (3.1), (3.2), $(5.4)_1$ and conditions leading to eqn (4.1) we obtain

$$S = x/G_B, \tag{5.5}$$

and

$$b^\nu_{n+1} = (-1)^n n! G_B \sum_{m=\bar{n}}^{n} \left(\frac{b}{2}\right)^{2m} \left(\frac{1}{m!}\right)^2 \sum_{k=\bar{n}}^{m} (-1)^{2m-k+n} \binom{m}{k} \binom{2k}{n} \left(\frac{x}{G_B}\right)^{2m-n}, \tag{5.6}$$

where $\bar{n} = n/2$ for even n and $(n+1)/2$ for odd n, $n \geq 0$ and

$$\binom{m}{k} = \frac{m!}{(m-k)!k!}.$$

The solution δ^ω to the posed time harmonic problem is given by eqns (3.1), (5.5) and (5.6).

The solution δ^H to the unit pulse problem pertaining to the boundary condition $(5.4)_2$ can be obtained from eqns (2.1) and (5.6) with $t_0 = x/G_B$. The time derivative of δ^H gives the Green's function or the unit impulse solution δ^Δ.

Closed form solutions δ^ω, δ^Δ of eqn $(4.1)_1$ for the above boundary conditions can be obtained by the method of separation of variables and Laplace transforms[12], respectively. They are

$$\delta^\omega = iG_B\, e^{i(kx-\omega t)}/(\omega^2 - b^2)^{1/2}, \qquad k^2 = (\omega^2 - b^2)/G_B{}^2, \qquad b^2 = G_K\alpha^{-1}, \tag{5.7}$$

$$\delta^\Delta = G_B J_0\{b^2 t^2 - x^2 b^2/G_B{}^2)^{1/2}\}, \tag{5.8}$$

where J_0 is the Bessel function of order zero. When eqn (5.7) is expanded in inverse powers of $(i\omega)$ and eqn (5.8) is expanded as a Taylor series about $t = x/G_B$, we obtain term by term agreement with our time harmonic and unit impulse solutions.

(iii) *A plate bending problem*

We consider a semi infinite plate which is given a transverse step velocity and has zero bending moment along the edge $x = 0$. The appropriate boundary conditions are

$$\frac{\partial u^3}{\partial t} = H(t), \quad \frac{\partial \delta^1}{\partial x} = 0, \quad \text{at } x = 0. \tag{5.9}$$

The boundary conditions (5.9) generate a first order kink wave and a second order bending wave. Setting $\mathbf{w}_2 = (\delta^1\nu, u^3)$, $A_{2n} = (-1)^n(b_n{}^\nu\nu, b_n{}^3)$ and using eqns (2.17), (2.19), (2.23) and (2.24) we can determine the discontinuities $(b_n{}^\nu, b_n{}^3)$ and $(\bar{b}_n{}^\nu, \bar{b}_n{}^3)$ pertaining to the kink ($G = G_K$) and bending ($G = G_B$) waves, respectively. Superposition of eqns (2.1) with $t_0 = x/G_K$ for the kink wave and $t_0 = x/G_B$ for the bending wave determines the required transient solution.

Boley and Chao in [13] considered this problem by dealing with the Timoshenko beam equations using Laplace transforms. In our calculations we set $\alpha_3 = 0.912\mu$, $\nu = 0.30$ to match their coefficient in the Timoshenko equations. In Fig. 2 we show the velocity $\dot{u}_3$ obtained from our solutions together with that obtained in [13]. The results have been nondimensionalised to correspond to the presentation in [13]. We observe the excellent agreement obtained with our series solution. We note that the modified superposition technique is not required here due to the short time range considered in the problem.

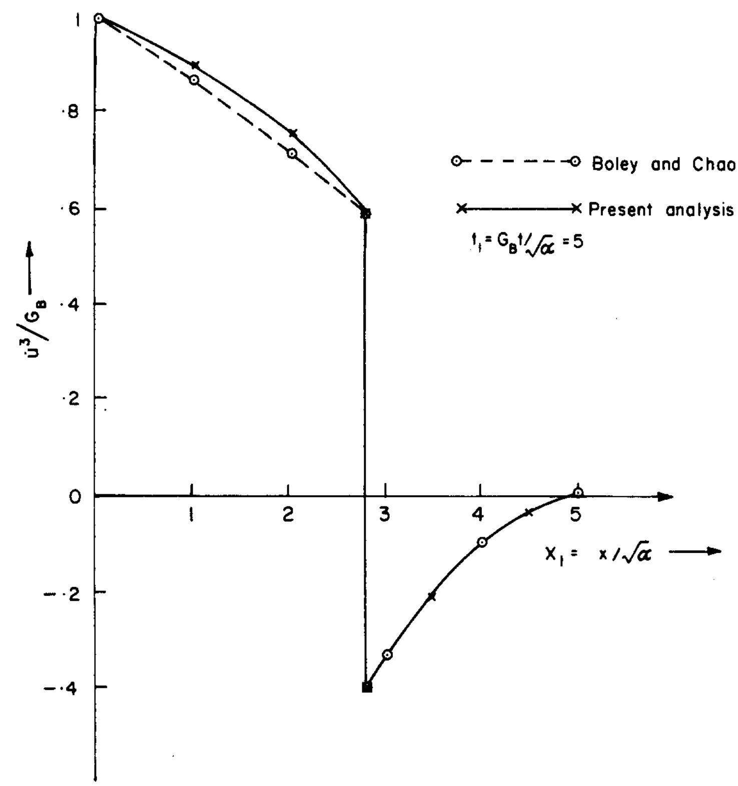

Fig. 2. Velocity distribution in the plate.

The faster moving bending wavefront and the kink wavefront are at $x_1 = 5$ and $x_1 = 2.8$ respectively. We observe that the velocity is continuous on the bending wavefront while it suffers a jump of unity (equal to the input at the boundary) in the kink wavefront.

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